Apollonius Revisited: Supporting Spheres for Sundered Systems

2 August, 1995

Victor Klee

University of Washington

O. Abstract

When S is the sphere that bounds a ball C in Euclidean $d-$space, we call S a "near support" for a convex body B provided that S intersects B but int C misses B. And S is a "far support" for B provided that S intersects B and C contains B. A main result asserts that if $(B_0, \dots, B_d)$ is a system of $d+1$ bodies that is, in a suitable sense, in general position, and (I, J) is an arbitrary partition of the index-set $\{0, ..., d+1\}$, then there is a unique sphere that is simultaneously a near support for B_i for all i in I and a far support for B_j for all j in J. When J is empty, the Brouwer fixed-point theorem provides an easy and elegant proof, but the general case involves more work.

There are also some definitive results on the behavior of supporting spheres for a system of d bodies in Euclidean $d-$space.

When the "spheres" themselves are permitted to be noneuclidean, problems concerning supporting spheres appear to be much more difficult. At present, complete results for the noneuclidean situation are available only for the 2−dimensional case.

(This is joint work with Ted Lewis and Balder Von Hohenbalken. It is related to an earlier seminar talk, but the results are now "definitive" for the Euclidean case, while earlier they were only fragmentary.)

Apollonius: Find all circles tangent to any given three in E^2.

Here, in R^2, $0 \leq \dots 8$ are possible, depending on the original three circles. But in higher dimensions, infinite families are possible, so we choose a sort of "general position": A system of bodies in R^d is sundered ($d-1$ separated) if, for each way of choosing a point from any $d+1$ of them, the points are affinely independent. (see also defn. last quarter).

A sphere S supports a body B if $S \cap B \neq \emptyset$ and $B \subseteq C$ or $B \cap (\text{int } C) = \emptyset$ where $S = \partial C$, C a ball. One can also speak of "far" $(B \subseteq C)$ and "near" $(B \cap \text{int } C = \emptyset)$ supports.

Main Theorem: $\forall$ Euclidean spheres, each sundered system $(B_0, \dots, B_d)$ in E^d has a unique $I-$near, $J-$far supporting sphere for B.

Corollary: Given a sundered system, the number of common supporting spheres is 2^m, where m is the number of non-singleton bodies.

Q: What happens when the spheres are non-Euclidean?

Idea: First prove a much weaker version assuming $\exists$ a unique sphere passing through each choice of points in the bodies, some of which are assumed to be balls (possibly degenerate to points). To do this, consider $\Pi_{i=0}^{d-1} B_i \subset R^{d(d-1)}$ and a function $\gamma(b)$ which is the center of that unique sphere corresponding to $b = (b_0, \dots, b_d)$. $\gamma(b)$ misses the center of every ball which is a body, so define $\sigma_{(I,J)} : \gamma(B) \rightarrow B$ by setting $\forall w \in \gamma(B)$, $\sigma_{(I,J)}(w) = \rho \in B$, where $\forall i \in I, \forall j \in J, \rho_i$ is the unique nearest point of B_i to w and ρ_j is the unique farthest point of B_j to w. σ is continuous, so it has a fixed point b which determines the desired $I-$near, $J-$far support.