The Genus of a Graph

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O. Abstract

Not available.

I. Combinatorial Embeddings

The Heawood conjecture is equivalent to the statement

$$\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$$

(and was proved by Ringel and Youngs in this equivalent form). One can define a combinatorial embedding by considering all possible permutations of the neighbor sets of each vertex, and defining a "face" of the embedding to be a clockwise walk. For example

Starting with edge (1,4), the successor to 1 is 2, so next take the edge (2,4). Then the successor to 4 is 6, so next take edge (2,6), and so on.

This gives a permutation set Π. γ_π is the combinatorial genus, and now the genus of the graph is simply $\gamma = \min_{\pi} \gamma_\pi$.

(See also the seminar report from Brian's previous talk on Snarks).

II. Genus of a Graph

By Kuratowski, we know every surface has a finite set of "forbidden subgraphs," but unfortunately as the genus increases, the size of the list increases greatly. Determining the genus of a graph is NP-complete.

The number of possible combinatorial embeddings of a graph G is $\prod_{v \in V}(deg(v)-1)!$. Let T be a spanning tree of G with rotational system (i.e., combinatorial embedding) Π. We say that $u, v \in E \backslash T$ overlap if $T \cup \{u, v\}$ is non-planar under Π. Let

$$A = (a_{i,j}) = \begin{cases} 1 & \text{if } e_i, e_j \text{ overlap} \\ 0 & \text{otherwise} \end{cases}$$

Then $\gamma_\pi(G) = \text{rank}(A)$ over $GF(2)$. This leads to a (new?) technique for finding the genus of a graph. Tinsley-Watkins used this to show that the genus of a flower snark is $\gamma(J_{2k+1}) = k$. In light of further work in this area, Brian has revised his former conjecture for the genus of products of the Petersen graph P to $1 \leq \gamma(P \cdot \cdots \cdot P) \leq 2(n-1)$, and that in fact each of these values of γ are realizable by some $n-$product of P.