Linear Conditions on f-Vectors of Polyhedra

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O. Abstract

In the theory of polytopes, it is well known that the Euler equations and the Dehn-Sommerville equations are the only (rational) linear conditions for f-vectors (numbers of simplices of various dimensions) of simplicial polytopes which can be viewed as triangulations of spheres. Following Klee's early paper on Euler manifolds, in this talk we consider f-vectors of triangulations of vectors of Euclidean triangulations and their linear conditions over Z and Z_k. For closed manifolds, the only linear condition is the Euler relation. For Eulerian manifolds with boundary index, some modulo integer relations do occur.

I. Preliminaries:

Let X be a $n-$dimensional polyhedron in R^n. Given a triangulation Δ on X, we have the $f-$vector $f=(f_0,f_1,\ldots,f_n)$ where $f_i=$ the number of $i-$dimensional simplices in Δ.

Problem: Classify all relations of the $f-$vectors for all triangulations of X. (usually we focus on linear relations)

II. Some results

Euler: $\sum_{k=0}^n f_k(-1)^k=\chi(X)$

Dehn-Sommerville for a $2-$dimensional manifold Σ^2,

$$f_0-f_1+f_2=\chi(\Sigma^2) \qquad 3f_2=2f_1$$

Klee classified polyhedra satisfying the Dehn-Sommerville relations, in a sense. (Recall the Dehn-Sommerville relations: If X is the bounary of a simplicial polytope, then all linear relations are generated by

$$f_i=\sum_{j=i}^{n-1}(-1)^{n-j-1}\left(\begin{array}{c} j+1 \\ i+1 \end{array}\right)f_j, -1\le i\le n-1$$

where $f_{-1}=1$ is the Euler relation. Only $\lfloor\frac{n}{2}\rfloor$ of these equations are independent.)

This work is stimulated by Klee's paper, *A combinatorial analysis of Poincare's duality theorem*, Canadian Journal of Math, 16 (1964) 517-531.

III. $f-$vectors of manifolds with boundary

Let σ be an open simplex in a triangulation Δ on M (with boundary ∂M). Fix $i=0,1,\ldots,n=\dim M$. Then (a sort of Möbius function):

$$
\begin{aligned}
\sum_{\sigma\le\tau,\dim\sigma=i}(-1)^{\dim\tau-\dim\sigma} &= \sum_{\sigma,\dim\sigma=i}(-1)^i\sum_{\sigma\le\tau}(-1)^{\dim\tau}\\
&= \sum_{\dim\sigma=i}(-1)^i\chi(\mathrm{nbd}(x)) \qquad \text{where } \chi \text{ is not the usual Euler number}\\
&\qquad \text{in the noncompact case, and } nbd(x) \text{ is a star-shaped}\\
&\qquad \text{neigborhood of an arbitrary point } x\in\sigma\\
&= \sum_{\dim\sigma=i}(-1)^i\begin{cases}(-1)^n & \sigma\subset M\backslash\partial M\\ 0 & \sigma\subset\partial M\end{cases}\\
&= (-1)^{n-i}f_i(M\backslash\partial M)\\
&= (-1)^{n-i}(f_i(M)-f_i(\partial M))
\end{aligned}
$$

On the other hand, the sum is also

$$
\begin{aligned}
&= \sum_{j\ge i}\sum_{\dim\tau=j}(-1)^j\sum_{\sigma\le\tau,\dim\sigma=i}(-1)^i\\
&= \sum_{j=i}^{n}\sum_{\dim\tau=j}\binom{j+1}{i+1}(-1)^{j-i}\\
&= \sum_{j=1}^{n}(-1)^{j-i}\binom{j+1}{i+1}f_j(M)
\end{aligned}
$$

so we conclude that [Dehn-Sommerville for general manifolds with boundary]

$$
f_i(\partial M) = \left[1-(-1)^{n-i}\right]f_i(M) + \sum_{j=i+1}^{n}(-1)^{n-j-1}\binom{j+1}{i+1}f_j(M)
$$

This can be thought of as a linear transformation from $Z^{n+1}\to Z^n$, where the $(n+1)\times n$ matrix is a "Dehn-Sommerville" matrix, which looks like

$$
D(n) = \begin{pmatrix}
x & -\binom{2}{1} & \binom{3}{1} & \cdots & \binom{n}{1} & -\binom{n+1}{1}\\
0 & 2 & \cdots & \cdots & \cdots & \cdots\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & \cdots & \cdots & 2\binom{n+1}{n}
\end{pmatrix}
$$

where $x=0$ if n is even, $x=2$ if n is odd.

We also get some theorems like $D(n)f(M,\Delta) = -D(n)f(M-\partial M,\Delta) = f(\partial M,\partial\Delta)$.

As a corollary, $D(n-1)D(n)=0$, $(I-D'(n))^2=I$ and

$$
\chi(D(n)v) = \begin{cases} 0 & n \quad \text{even}\\ 2\chi(v) & n \quad \text{odd}\end{cases}
$$

where $\chi(v)=\sum_{k=0}^{n}v_k(-1)^k$.

[He considered the rational case, using a technical lemma concerning the existence of polytopal (Euclidean) triangulations δ_i with $\{f(D^n, \delta_i) - f(D^n, \delta_0)\}$ linearly independent, $1 \le i \le \lfloor \frac{n+1}{2} \rfloor$. See also Brønsdted, *Intro. to Convex Polytopes.* He found that the rational affine span $f(M,$ all) is generated by the Euler equation, while the integer affine span $f(M \text{rel } T,$ all) is generated by Euler and Dehn-Sommerville.]

IV. Euler manifolds with boundary (c.f. Klee)

A polyhedron pair $(X, \partial X)$ is an Euler manifold with boundary if $\exists \beta \neq 0$, an integer such that

$$\chi(\text{link}(x, X)) = \begin{cases} 1 - (-1)^n & x \notin \partial X \\ 1 - (-1)^n + (-1)^n \beta & x \in \partial X \end{cases}$$

β is called the boundary index. If $\beta = 0$, X is an Euler manifold (equivalent to Klee's definition).

For example, $S^2 \vee S^2$ (the one-point union of 2 spheres) is an Euler manifold – the link at that point is two circles. $S^2 \vee S^4$ is also. $S^2 \vee S^1$ is an Euler manifold with $\beta = 2$. Every manifold with boundary is an Euler manifold with $\beta = 1$.

Proposition: Let Δ be a triangulation of $(X, \partial X)$. Then $(X, \partial X)$ is an Euler manifold with boundary index β if $\forall$ simplices $\sigma \in \Delta$,

$$\chi(\text{link}(\sigma, \Delta)) = \begin{cases} 1 - (-1)^{n-\dim\sigma} & \sigma \notin \partial X \\ 1 - (-1)^{n-\dim\sigma} + (-1)^{n-\dim\sigma} \beta & \sigma \in \partial X \end{cases}$$

$$(1 - (-1)^n)\chi(X) = \beta\chi(\partial X)$$

Corollary: If X is a compact Euler manifold without boundary, then $\chi(X) = 0$

[He also gave a technical theorem, $(X, \partial X)$ is an Euler manifold with boundary value β iff exactly one of three conditions holds..., etc.]

Theorem: $\partial(X, \partial X)$ is an Euler manifold without boundary. $\dim X + \dim \partial X$ is odd.

Theorem: If $(X^n, \partial X)$ is a compact Euler manifold with boundary index β, then the integer affine space of $(X, \text{rel}T)$ can be characterized by $\chi(c) = \chi(x)$ and $D(n)v = \beta f(\partial X, T)$. And if $\partial X \neq \emptyset$, boundary dimension is γ, then the integer affine span $f(X)$ is characterized by $\chi(v) = \chi(x)$, $[D(r+1), F(r+1, n)]v = 0 (\text{mod } \beta)$, and $[0, E(r+1, n)]v = 0$, where

$$D(n) = \begin{bmatrix} D(m) & F(m,n) \\ 0 & E(m,n) \end{bmatrix}$$