Realizations of Regular Maps

9 August, 1995

Heidi Burgiel

University of Washington

O. Abstract This talk will deal with polyhedral embeddings of regular maps and compounds of regular maps of type $\{4,4\}$, $\{3,6\}$, and $\{6,3\}$ in which all automorphisms of the maps are realized as symmetries of the embedded polyhedra. If time permits, the talk will include methods for doing this which can also be applied to more general regular maps.

I. Maps

A **map** is just like what one would expect – non-intersecting edges, simply connected faces, etc. A **flag** is a $\{v, e, f\}$ triple. A map is **regular** if its automorphism group acts transitively on its flags. Platonic solids are the analog of regular maps on S^2. Square maps of type $\{4,4\}_{b,0}$ are skew polyhedra $\{b\} \times \{b\}$. [Petrie, circa 1926]

A **regular polygonal map** is the decomposition of a $1-$manifold without boundary (loop or line) into overlapping simply connected edges joining vertices. A **regular skew polygon** is a collection of vertices and edges in E^d such that the automorphism group corresponds to some regular polygonal map. (skew, because the vertices need not all lie in a plane) A **regular skew polyhedron** is a collection of $\{v, e, f\}$ triples in E^d such that incidence relations in P correspond to those of some regular map M, and P's symmetry group corresponds to the automorphism group of M.

Example: Realize a square grid on the torus T^2 by $(x, y) \to (e^{ix}, e^{iy}) \in R^4$.

II. Representations

A **faithful representation** is one in which the correspondence between vertices of P and M are one-to-one. If M has finitely many vertices, a representation (the simplex representation) with each vector in M taken to the unit vector in R^n works. For infinite vertex sets, the problem of finding faithful representations is unsolved. Goal: Find low-dimensional faithful realizations of finite regular maps.

The permutation representation of M is found by treating $Aut(M)$ as a permutation group on the vertices of the simplex realization $S(M)$. One can think of "shadows" of $S(M)$ on lower-dimensional subspaces fixed by $Aut(M)$. Then the images of $S(M)$ under the projections to these lower-dimensional subspaces will be **irreducible.**

Let $A = \{M \in GL^n(R) | MgM^{-1} = g \ \forall g \in G = Aut(M)\}$ be the **centralizer algebra** of G. If the map is "nice" [paths in the edge-graph commute – i.e., irreducible components are multiplicity-free], then the spaces fixed under $\mathcal{A}$ are precisely those fixed by $Aut(M)$, and $\mathcal{A}$ has a basis.

Example: $M_{4,5}$ has vertices $\{v_1, \ldots, v_n\}$. To form a basis for $\mathcal{A}$ that commutes with the elements of $Aut(M)$, define A_i by

$$(A_i)_{j,k} = \begin{cases} 1 & \text{if } v_j \text{ is distance } i \text{ from } v_k \\ 0 & \text{otherwise} \end{cases}$$