Sphere Packing in Dimensions 1-24

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O. Abstract

Not available

I. Volume of the unit sphere

The volume of the unit sphere $||x|| \leq 1$ in the Euclidean norm can be found, for example, by integration, which gives

$$V_n = \int_{-1}^{1} V_{n-1}(1-x^2)^{n-1}dx$$

which can be integrated using beta and gamma functions. An alternative method, however, is to use a neat trick: Consider

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2+\cdots x_n^2)}\ dx_1 dx_2 \cdots dx_n \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^n \\ &= \pi^{n/2} \end{aligned}$$

but on the other hand,

$$\begin{aligned} I &= \int_{x \in R^n} e^{-|x|^2} dV \\ &= \int_{x \in R^n} \left(\int_{y \in R,\ y \geq |x|^2} e^{-y} dy\right) dV \\ &= \int_0^{\infty} \left(\int_{|x|^2 \leq y} e^{-y} dV\right) dy \\ &= \int_0^{\infty} e^{-y} y^{n/2} V_n dy \\ &= V_n\ \Gamma(1+\frac{n}{2}) \end{aligned}$$

so

$$V_n = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}$$

II. Sphere packing densities

Let $\delta(S_{n-1})$ = the maximum sphere-packing density in R^n. Then $\delta(S_1) = 1$ (obvious), $\delta(S_2) = \pi/2\sqrt{3}$ (Thue, 1910), and $\delta(S_3) \stackrel{?}{=} \pi/3\sqrt{2}$ (Hsiang, but it's not clear if it's correct). A related quantity is Roger's bound (1958), that $\sigma_n \geq \delta(S_n)$. [Incidentally, $\sigma_n \sim \frac{n}{e} 2^{-n/2}$.] This has been improved to $\delta(S_3) \leq .778\overset{?}{4}\ldots$ by Lindsay, 1986, and Muder 1988.

This problem is so hard that one generally restricts to the case that the centers lie on a lattice. $\delta_L(S_n)$ is known up through (and including) the case $n = 8$.

We can use the integer span of the columns of a matrix to generate a lattice, iff we have a positive definite quadratic form f such that $\det f = \det(B^T B)$ and $M(f) \cong \min_{x \in Z^n \backslash 0} f(x)$

Define $\gamma_n = \max_f \frac{M(f)}{\sqrt[n]{d(f)}}$. Since it is known that $.5\sqrt{\gamma_n}\, V_n = \delta_L(S_n)$, quadratic forms play a role in studying sphere-packing densities.

Through $n = 8$, and probably $n = 10$, one can create the maximum lattice packing by "lamination" (adding a unit vector to the packing before it). Unfortunately, this fails for $n = 11$. Define Λ_0 to be a one-point lattice, and Λ_n to be the nth "laminated" lattice. Then through $n = 24$, these are unique except for $n = 11, 12, 13$. Indeed, except for these three cases, the densest known lattice packings are Λ_n! Cases 13 through 24 (except for 16, which had already been settled by Barnes and Wall in 1959) were found by Leech in 1965 as cross-sections of the famous Leech lattice Λ_{24}. (See also Conway's paper; one construction is to take the binary word 11101101000, make a matrix of all left shifts of it, extend, and obtain the binary Golay code it generates. Then take an orthogonal basis $\{b_i\}$ for R^{24} and let Λ_{24} be all linear combinations $\sum_i t_i b_i$ such that either all the t_i are even and $\sum t_i \cong 0 \pmod 8$ and $.5(\sum t_i \pmod 2)$ is in S, or else all are odd, $\sum t_i \cong 4 \pmod 8$, and $.5 + .5(\sum t_i) \pmod 2)$ is in S.)

In some cases, it is known there exist better packings with nonlattices. What is the general situation?