Remarks on the Graceful Tree Labeling Conjecture

Let $[0,n] = \{0,1,2,\cdots,n\}$ and let T be a tree with n edges. A function g is called a graceful labeling of T is g is an injection from the vertices of G to the set $[0,n]$ such that the values $|f(x) - f(y)|$ for the n pairs of adjacent vertices x, y are distinct.

For each tree T with n edges we will show the existence of a multivariable polynomial $f_T(\overrightarrow{y})$ ($\overrightarrow{y} \in R^{n^2+n}$) with the property that $\boxed{T \text{ has a graceful labeling if and only if } f_T(\overrightarrow{y}) \not\equiv 0}$ (the zero polynomial).

Label the vertices of T with the indeterminates $x_0, x_1, \cdots, x_n$– from now on we will identify a vertex by its indeterminate. Arbitrarily assign a direction to each edge in T (this gives us a directed tree $\overrightarrow{T}$). Assume that $\overrightarrow{e}_i = (x_{i_1}, x_{i_2})$ $1 \le i \le n$ are the n directed edges of $\overrightarrow{T}$. Let $p_T(x_0, x_1, \cdots, x_n) = \Pi_{n \ge i > j \ge 1} \left(((x_{i_2} - x_{i_1}) - (x_{j_2} - x_{j_1})) \cdot ((x_{i_2} - x_{i_1}) - (x_{j_1} - x_{j_2}))\right)$ and let $f'_T(x_0, x_1, \cdots, x_n) = \Pi_{n \ge i > j \ge 0}(x_i - x_j) \cdot p_T(x_0, x_1, \cdots, x_n)$. Let $D = [0,n]^{n+1}$. Clearly T has a graceful labeling if and only if $f'_T : D \to Z$ is non-zero at some point in D.

Now replace each x_i in f'_T by $\sum_{j=1}^{n} y_{i,j}$. Thus f'_T now becomes a new polynomial in the indeterminates $y_{0,1}, y_{0,2}, \cdots, y_{0,n}, y_{1,n}, \cdots, y_{n,1}, \cdots, y_{n,n}$. Call this new polynomial f_T. Let $Q^{n^2+n} = \{0,1\}^{n^2+n}$(i.e. Q^{n^2+n} is the n^2+n dimensional hypercube). It is not difficult to see that T has a graceful labeling if and only if $f_T : Q^{n^2+n} \to Z$ is non-zero at some point in Q^{n^2+n}. [(The presence of $\Pi_{n \ge i > j \ge 0}(x_i - x_j)$ as a factor in f'_T should help convince the reader of this fact.)]

Recall that a polynomial in several variables is multilinear if its degree in each variable is at most 1. Since the domain of f_T is restricted to $Q^{n^2+n} \subseteq R^{n^2+n}$ (here $x_i^2 = x_i$) we may assume that f_T is multilinear.

Now we will make use of the following lemma.

Lemma 1 (Alon and Tarsi []). Let $P = P(x_0, x_1, \cdots, x_K)$ be a polynomial in $K+1$ variables over an arbitrary field F. Suppose that the degree of P as a polynomial in x_i is at most c_i for $0 \leq i \leq k$, and let $A \subseteq F$ be a set of cardinality $c_i + 1$. If $P(x_0, x_i, \cdots, x_K) = 0$ for all $(K+1)$-tuples $(x_0, \cdots, x_k) \in A_0 \times A_1 \times \cdots \times A_K$, then $P \equiv 0$.

Since f_T is multilinear, we see that T has a graceful labeling if and only if $f_T \not\equiv 0$.

Therefore we have proved the following claim.

Claim. For every tree T with n edges there exists a polynomial $f_T(\vec{y})(\vec{y} \in R^{n^2+n})$ Such that T has a graceful labeling if and only if $f_T(\vec{y}) \not\equiv 0$.

H. Snevily